

# Almost Diameter Rigidity for Cayley Plane

Akhil Ranjan

February 7, 2008

## Abstract

In this paper we give a generalisation of the Radius Rigidity theorem of F. Wilhelm. This is done by showing that if a Riemannian submersion of  $S^{15}$  with 7-dimensional fibres has at least one fibre which is a great sphere then all the fibres are so. Some weaker than known conditions which force the existence of such a fibre are also discussed.

**Keywords:** Curvature, Diameter, Riemannian submersions, Isoparametric foliations.

## 1 Introduction

Gromoll and Grove in their beautiful papers [2] and [3] almost classified the Riemannian submersions of the round spheres and applied it to strengthen M. Berger's rigidity theorem characterising the compact rank one symmetric spaces. They termed it as the Diameter Rigidity Theorem. The analysis done in [2] showed that the hypotheses involved in the Diameter Rigidity Theorem gave rise to a Riemannian submersion of the unit tangent sphere at some point onto its cut-locus. The classification of these submersions then enables one to prove symmetry. However, just one case, that of Cayley-plane remained unsolved as the corresponding classification of Riemannian submersions of  $S^{15}$  with 7-dimensional fibres remained unsettled. Given such a submersion, F. Wilhelm has

---

<sup>0</sup>AMS Subject Classification: 53C35

<sup>0</sup>MRR 07-97

shown that the set of points in the base over which the fibres are great spheres, has strong geometric properties (Main Lemma in [7]) and in conjunction with the assumed lower bound on the radius shows that all the fibres, in the Riemannian submersion which arises, as in [2] are great spheres. This is what precisely is missing if only diameter condition is assumed.

In the present work it is shown that Wilhelm's Main Lemma can be considerably improved and consequently the following **Almost Diameter Rigidity** can be proved.

**Theorem 1.1** *If a nonspherical Riemannian manifold  $M$  has sectional curvatures  $\geq 1$  and has an equilateral triangle of sides  $\pi/2$ , then it must be a symmetric space.*

In this paper we will only consider the case where  $M$  is a cohomology  $CaP^2$ .

It is interesting to note that the above can also be compared with *corollary II* of [1] where it is required that every pair of points  $\pi/2$  distance apart be completed into an equilateral triangle.

It is indicated towards the end of the paper how one can further weaken a little even this equilateral triangle condition.

To prove the above theorem we prove the following very strong version of Wilhelm's Main Lemma

**Theorem 1.2** *Given a Riemannian submersion of  $S^{15}$  with 7-dimensional fibres, the set of those points in the base manifold  $B$  over which the fibres are great spheres, is either empty or all of  $B$ .*

## 2 Notations and Preliminaries

Let us recall some standard notations and results used in this paper. Let  $S^{n+k}$  be the sphere of constant sectional curvature, the constant being assumed to be unity. Let it be fibred by leaves of dimension  $k$  so that the metric is "bundle-like" on the sphere or in other words we have a Riemannian submersion onto a Riemannian manifold  $B$ . O'Neill [4] introduced tensors  $T$  and  $A$  for any metrically foliated manifold  $M$ , and called them *second fundamental tensor* and *integrability tensor* respectively. First of all one decomposes the tangent bundle into a direct sum  $\mathcal{V} \oplus \mathcal{H}$  where  $\mathcal{V}$  is the set of vectors

tangential to the leaves and  $\mathcal{H}$  are those normal to the leaves. They are called vertical and horizontal vectors respectively. For any tangent vector  $e$ , we write  $e = e^v + e^h$  or sometimes  $\mathcal{V}(e) + \mathcal{H}(e)$ , uniquely as a sum of vertical and horizontal vectors. For  $U$  and  $V$  smooth vertical vector fields and  $X$  and  $Y$  horizontal, we define

$$A_X Y = (\nabla_X Y)^v, \text{ and } A_X V = (\nabla_X V)^h \quad (2.1)$$

$$T_V U = (\nabla_V U)^h, \text{ and } T_V X = (\nabla_V X)^v \quad (2.2)$$

Here  $\nabla$  denotes the Levi-Civita connection on the manifold. For any horizontal vector  $x$  and vertical vector  $v$  at a point  $p \in M$ ,  $A_x$  and  $T_v$  are skewsymmetric endomorphisms of  $T_p M$  which interchange vertical and horizontal vectors. Moreover we have

$$A_x y = -A_y x, \text{ and } T_u v = T_v u \quad (2.3)$$

For convenience we also write

$$A^v x = A_x v, \text{ and } T^x v = T_v x \quad (2.4)$$

This makes  $A^v$  (for a vertical  $v$ ) act as a skewsymmetric operator on horizontal vectors while  $T^x$  (for  $x$  horizontal) becomes a symmetric endomorphism of vertical vectors. The latter is written as  $S_x$  in [3]. It is the second fundamental form operator.

Finally, we make  $\nabla^v$  denote the (Riemannian) connection on  $\mathcal{V}$  obtained by vertical projection of  $\nabla$  and  $\nabla^h$  that on  $\mathcal{H}$ . Also as in [4], a horizontal vector field  $X$  which projects to a well defined vector field in  $B$ , will be referred to as *basic*. It satisfies the differential equation

$$\nabla_V^h X = A_X V = A^V X \quad (2.5)$$

where as usual  $V$  is any vertical field.

### 3 Some Computations

**Lemma 3.1** *Let  $X$  and  $Y$  be basic fields along a fibre of a Riemannian submersion then*

$$\text{div}(A_X Y) = - \sum_i < (\nabla_{v_i} A)^{v_i} X, Y > \quad (3.6)$$

*In the above equation,  $\text{div}$  denotes the divergence operator in the fibre and accordingly  $\{v_i\}$  is an orthonormal basis of vertical vectors at any point of the fibre where divergence is to be found.*

**Proof:** Straightforward. The terms involving derivatives of  $X$  and  $Y$  cancel out on invoking skew-symmetries of  $A$ . ■

**Corollary 3.1** *In the case of a Riemannian submersion of the round sphere, the fields  $A_X Y$  are divergence free for  $X$  and  $Y$  basic along a fibre.*

**Proof:** From equation {2} of [4] and using isoparametricity [3] we easily see that  $\text{div}(A_X Y)$  is constant along the fibre. Since average value of divergence of a tangent vector field on a closed manifold is zero, we get the result. ■

**Remark:** Isoparametricity shows that the terms  $\langle T^X, T^Y \rangle$  and  $\langle \nabla_X K, Y \rangle$  are constant along the fibre, where  $K$  is the mean curvature vector field. For a simple geometric proof of isoparametricity one can also see [6]. Also the constant sectional curvature implies that  $|A_X Y|^2$  is constant along the fibres for basic fields  $X$  and  $Y$  (see [3] or use eqn.{4} of [4]).

**Corollary 3.2** *Let  $X$  be a basic field along a fibre of a Riemannian submersion of the round sphere, then*

$$\Delta X = \sum_i (A^{v_i})^2 X. \quad (3.7)$$

Here  $\Delta$  is the Laplacian computed by using the (Riemannian) connection  $\nabla^h$  on  $\mathcal{H}$  restricted to the fibre.

**Proof:**

$$\Delta X = \sum_i [(A^{v_i})^2 X + (\nabla_{v_i} A)^{v_i}(X)]^h = \sum_i (A^{v_i})^2 X \quad (3.8)$$

■ Now in the case of spheres the operator  $\sum_i (A^{v_i})^2$  actually descends to the base space  $B$  and defines a symmetric negative definite endomorphism of the tangent space  $T_b B$  at each point  $b \in B$ . This is due to last statement in the remark above. If we fix one such  $b$  over which our fibre under consideration lies, we see that one can get an orthonormal framing of the fibre by basic fields  $\{X_1, X_2, \dots, X_n\}$  which are eigen vectors of the operator  $\sum_i (A^{v_i})^2$  and hence also of  $\Delta$ .

## 4 Proofs of the Theorems

We will be restricting ourselves to the case of a Riemannian submersion of  $S^{2k+1}$  onto  $B^{k+1}$  with  $k$  dimensional fibres one of which, say  $F$ , is a great sphere. In this case

its orthogonal complement  $F^\perp$  is also a great sphere and a fibre. We have a parallel orthonormal framing  $\{E_0, E_1, \dots, E_k\}$  of the horizontal vectors along  $F$ . Let  $\{X_j, 0 \leq j \leq k\}$  be the basic framing by the eigen vectors of the Laplacian and let, for simplicity of notation,  $X$  be any one of these with eigen value  $-\lambda$ . Clearly, one can write  $X = \sum_j f_j E_j$  for suitable uniquely determined smooth functions on  $F$ . This easily yields that each  $f_j$  is an eigen function of the Laplacian with the same eigen value  $-\lambda$ . Thus each  $f_i$  is a spherical harmonic in  $k+1$  variables and  $\lambda$  is in the spectrum of  $(S^k, g_{can})$ .

Now fix this basic eigen vector field  $X$  along  $F$ . Let  $\{V_i, 1 \leq i \leq k\}$  be an orthonormal framing of  $TF$  by eigen vectors of the operator  $A_X^2$  acting on vertical vectors. This can be done as follows. As  $T^X$  vanishes,  $A_X$  is injective on  $\mathcal{V}$  (see [3],[6]) and therefore nonsingular on  $X^\perp$  due to dimension restrictions. Hence can find a basic  $k$ -frame  $\{Y_i\}$  such that  $A_X^2 Y_i = -\mu_i^2 Y_i$  with  $\mu_i > 0$ . Set  $V_i = A_X Y_i / \mu_i$ . Trivially,  $A_X^2 V_i = -\mu_i^2 V_i$ .

**Lemma 4.2** *As an operator on the vertical space at any point  $p \in F$ , trace of  $A_X^2$  satisfies the inequality*

$$tr A_X^2 \leq -k$$

.

**Proof:** Let  $\gamma$  be a horizontal geodesic starting at  $p$  in the direction  $X(p)$ . Let  $v_i$  denote the parallel translate of  $V_i(p)$  along  $\gamma$  and  $e_i$  that of  $E_i = A_X V_i(p)$ . Then the horizontal holonomy displacement of  $V_i(p)$  along  $\gamma$  is given by

$$V_i(t) = \cos t v_i(t) + \sin t e_i(t)$$

In particular,  $V_i(\pi/2)$  is a basis of  $T_{\gamma(\pi/2)} F^\perp$ . Consider the matrix

$$M(t) = ((\langle V_i(t), V_j(t) \rangle)) = \cos^2 t I + \sin^2 t (\mu_i^2 \delta_{ij}) = \text{diag}(\cos^2 t + \mu_i^2 \sin^2 t)$$

. Due to isoparametricity,  $\sqrt{\det M(t)}$  gives the ratio of the volume of the fibre  $F(t)$  through  $\gamma(t)$  and that of initial fibre  $F$ . Since  $F$  is isometric to  $F(\pi/2) = F^\perp$  we get

$$\prod_i \mu_i = 1$$

This implies  $\sum_i \mu_i^2 \geq k$  as claimed. ■

**Lemma 4.3** *Let  $X = \sum_i f_i E_i$  for suitable harmonic polynomials  $\{f_i\}$  in  $k+1$  variables  $\{u_i, 0 \leq i \leq k\}$  and parallel orthonormal frame  $\{E_i\}$  for any basic eigen vector field  $X$  along  $F$ , then each  $f_i$  is linear.*

**Proof:**  $\{u_i\}$  are coordinates in the  $\mathbb{R}^{k+1}$  spanned by  $F$ . Likewise, let  $\{v_i\}$  denote the remaining coordinates in the Euclidean space of  $F^\perp$ . The horizontal holonomy map from  $F$  to  $F^\perp$  induced by geodesics starting from various points of  $F$  in the direction of  $X$  is given by

$$v_i = f_i(u_0, \dots, u_k), 0 \leq i \leq k$$

Thus it is a diffeomorphism which is algebraic. We will see presently that its inverse is also algebraic. The holonomy displacement produces a basic field  $Y$  along  $F^\perp$  and by the same considerations as for  $F$ , it is a restriction of a polynomial vectorfield i.e. the coefficient functions with respect to a parallel orthonormal framing of  $\mathcal{H}$  are polynomials  $g_i(v_0, \dots, v_k)$ . But the holonomy displacement from  $F^\perp$  to  $F$  via  $Y$  is same as that given by  $X$  at  $t = \pi$ . Hence  $g_i(f_0, \dots, f_k) = -u_i$ . This proves that  $f_0, \dots, f_k$  give an algebraic equivalence. It follows therefore that  $\sum_i f_i^2 - 1$  should generate the ideal  $(\sum_i u_i^2 - 1)$  and this forces  $f_i$  to be linear homogeneous. ■

**Corollary 4.3** *As an operator on vertical vectors,*

$$A_X^2 = -I$$

*for any unit horizontal vector  $X$  at any point of  $F$ .*

**Proof:** Since the coefficients of any basic field along  $F$  are linear it follows that  $\lambda_i = k$  for every  $i$ . Therefore,  $\sum_i (A^{v_i})^2 = -kI$  and consequently for any unit horizontal vector at any point of  $F$ ,  $\sum_i |A_X v_i|^2 = k$ . This is same as  $\text{tr} A_X^2 = -k$  and this in turn forces  $\mu_i = 1$  for each  $i$ . Clearly then  $A_X^2 = -I$ . ■

**Proof of Theorem 1.2:** From the above corollary we see that  $\sqrt{\det M(t)} = 1$  for all  $t$ . Hence volume of  $F(t)$  is constant and equal to that of  $F(0) = F$ . Thus every fibre is of same volume as  $\text{vol}(F)$  and therefore again by isoparametricity, each fibre is a great  $k$ -sphere. Now it follows from the results proved in [5] that the submersion is congruent to the Hopf fibration. ■

**Proof of Theorem 1.1:** Let  $x$ ,  $y$ , and  $z$  be mutually distance  $\pi/2$  apart. In this situation  $y$  and  $z$  both are in the dual set  $\{x\}'$  of  $x$ . (See [2],[7] for details about dual sets.) From [2] we know that there is a Riemannian submersion

$$\exp_x : S_x \rightarrow \{x\}'$$

with 7-dimensional fibres from the unit tangent sphere at  $x$ , and similarly for  $y$  and  $z$ . As argued in [7] this forces at least one fibre to be totally geodesic in each case.

But then  $\exp_x$  is congruent to the Hopf fibration and this makes the space isometric to  $(CaP^2, g_{can})$  as remarked in [2].

## 5 Concluding Remarks

Due to isoparametricity of Riemannian submersions of spheres, it is easy to see that the procedure of averaging the smooth functions over the fibres projects any eigen-space of the Laplacian orthogonally into itself. Now the question which arises is whether the spherical harmonics of degree two admit a nonzero function which is constant along the fibres or not. If yes then the critical sets being great spheres, we get smaller saturated great spheres. In case of  $S^{15}$ , if there is a fibre trapped in the zone  $\mathbb{R}^{15} \times [-1/4, 1/4]$ , then one can show that the harmonic  $x^2 - 1/16$  can be averaged to give a nonzero harmonic,  $x$  being the last coordinate in the above decomposition. Hence there exists a proper great sphere which is a union of fibres. Due to topological restrictions, in case we are dealing with 7-dimensional fibres, it can only be a single fibre. Thus the Main Theorem is further strengthened to

**Theorem 5.3** *If a Riemannian submersion of  $S^{15}$  is given with 7-dimensional fibres, and there is a fibre trapped in the zone  $\mathbb{R}^{15} \times [-1/4, 1/4]$  then it must be congruent to a Hopf fibration.*

**Corollary 5.4** *If  $S^{15} \rightarrow B^8$  is a Riemannian submersion such that the diameter of the base  $B$  is at least  $\cos^{-1}(1/4)$ , then it must be congruent to the Hopf fibration.*

Now Theorem 1.1 can be rephrased suitably with a weaker hypothesis.

## References

- [1] O. Durumeric, *A Generalization of Berger's Theorem on Almost 1/4 Pinched Manifolds II*, J. Diff. Geom. **26**(1987), 353-375.
- [2] D. Gromoll and K. Grove, *A Generalization of Berger's Rigidity Theorem for Positively Curved Manifolds*, Ann. Sci. Ecole Norm. Sup., 4 serie, t. 20 (1987), 227-239.

- [3] D. Gromoll and K. Grove, *The low Dimensional Metric Foliations of Euclidean Spheres*, J. Diff. Geom., **28**(1988), 143-156.
- [4] B. O'Neill, *The Fundamental Equations of a Submersion*, Mich. Math. J., **13**(1966), 459-469.
- [5] A. Ranjan, *Riemannian Submersions of Spheres with Totally Geodesic Fibres*, Osaka J. Math. **22**(1985), 243-260.
- [6] A. Ranjan, *Metric Foliations of Spheres*, Conference on Geometry and Topology, IISc., Bangalore (Dec. 1996).
- [7] F. Wilhelm, *The Radius Rigidity Theorem for Manifolds of Positive Curvature*, J. Diff. Geom., **44**(1996), 634-665.

Department of Mathematics  
 Indian Institute of Technology  
 Mumbai 400076, INDIA  
 email: aranjean@ganit.math.iitb.ernet.in